Entropy closure model for transport in Hall thruster simulations

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Abstract: In this paper, we introduce a new approach to treating the electron transport in Hall thrusters. The overarching objective is to find an electron transport model that can be imported into simulations to guide in the engineering development of new Hall thruster configurations and operating conditions. In the approach here, we model the electron entropy production and its scaling with effective collision frequency and magnetic field and complement the usual equations for the electrons with a transport equation for electron entropy. This additional equation allows us to close the set of equations for the electron fluid without the need to have to introduce an electron mobility. The electron mobility becomes a calculated parameter within the framework of these simulations. Initial results are presented for the calculation of the electron mobility for operating conditions of a laboratory Hall thruster as well as for an SPT-100.

Nomenclature

α	=	entropy production parameter
B	=	magnetic field
c_e	=	mean electron speed
e	=	electron charge
E	=	electric field
h	=	Planck constant
k_B	=	Boltzmann constant
K_{eff}	=	effective electron thermal conductivity
m_e	=	electron mass
$\hat{n}_\perp$	=	unit normal vector along direction perpendicular to the magnetic field contour
μ_{eff}	=	effective electron mobility
n_e	=	electron number density
$\dot{n}_e$	=	net volume rate of electron production
$\vec{q}_e$	=	electron heat flux
s_e	=	electron entropy
$\dot{s}_e$	=	electron entropy production rate per unit volume

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T_e	=	electron temperature
$\vec{u}_e$	=	electron drift velocity
ν_{eff}	=	effective electron scattering frequency
ω_{ce}	=	electron gyrofrequency

I. Introduction

PHYSICAL electron transport models for Hall thruster simulations have been proposed recently in an attempt to establish a more general description of the electron migration that is more widely applicable over broad ranges of operating conditions[1,2]. Among these are models that introduce localized transport barriers attributed to shearing in the electron fluid. These transport barrier models have been incorporated into two-dimensional hybrid thruster simulations[2-4]. However, most of these models still require some parameterization and in some cases, their implementation has not been straightforward. An exception, perhaps, is the isentropic electron model[5], although that model was found to work well for only a limited set of conditions and only in a one-dimensional fluid simulation. Attempts to implement that model into hybrid simulations[6] have been marginally successful[7].

In this paper, we discuss a new approach to modeling transport in Hall thrusters. This approach is motivated by the promising results of Ref. 5 which introduce the entropy equation as a possible means of closing the set of electron fluid equations thereby releasing the need to specify the electron mobility. In the study of Ref. 5, it was assumed that the entropy production in the electron fluid is negligible (i.e., the electron fluid is isentropic in its behavior). We believe that in some regions of the flow (where there is little collisionality and a high magnetic field, for example) this may be a reasonable approximation; however, in regions where there is a significant electron scattering the assumption of isentropic electron flow is not valid. In this new approach, we instead model the *entropy production rate*. With this term added to the electron entropy transport equation the electron mobility becomes a calculated parameter that is dependent on the local properties in the electron fluid.

II. Theory

Our approach to modeling entropy production is to examine through its scaling the possible dependence on important properties of the plasma such as the plasma density, plasma collisionality, and the applied magnetic field. In this way, we use dimensional reasoning[8] combined with some guidance from experiments to arrive at the most primitive description of the local scaled rate of volumetric entropy production, $\dot{s}_e/k_B$, in the electron fluid. The relevant plasma parameters that are expected to factor in this entropy production rate include the local plasma density, n_e, the local magnetic field, B (through the local electron cyclotron frequency $\omega_{ce} = eB/m_e$), and the local effective collision frequency, ν_{eff}, which encompasses both physical and virtual electron scattering, i.e., the scattering of electrons as a result of coherent and turbulent fluctuations in the electric field. Such scattering is important in entropy production as it allows the electrons to sample the micro-canonical electron energy states of the system. The experiments guide us in this reasoning through the wealth of data that suggests that in regions of strong magnetic field, the effective collisionality (inverse Hall parameter) is weak and so the rate of entropy production is expected to be small in this region.

The dimensional reasoning reduces the maximal set of non-dimensional variables to two[9], which leads to the dependence of the entropy production on an unknown function, *f*, of the Hall parameter, ω_{ce}/ν_{eff},, i.e.,

$$\dot{s}_e = n_e k_b \nu_{eff} \cdot f\left(\omega_{ce}/\nu_{eff}\right) \tag{1}$$

Of course, this functional dependence on the Hall parameter is not known, and for a maximal set of parameters greater than unity, must, in general, be determined by empirical means or perhaps a more detailed theory. Using experiments to guide us in this regards we recognize that in regions of strong magnetic fields, i.e., near the exit plane of a typical Hall thruster, several researchers[10,11] have found (see, for example, Fig. 1, taken from Ref. 11) that there is a strong transport barrier that results in an effective collision frequency that is very low (and often seems to approach classical values which are very low). The ad-hoc introduction of this transport barrier into Hall thruster modeling[2-4] has resulted in simulations that describe many features of Hall thruster operation, including performance. The results suggest, then, that in the limit of large values of ω_c/ν_{eff}, the function $f\left(\omega_c/\nu_{eff}\right) \to 0$. At this time we test the simplest of functions – one that is linear in the inverse Hall parameter, i.e.,

$$f\left(\omega_c / \nu_{eff}\right) \approx \alpha \cdot \nu_{eff} / \omega_c \tag{2}$$

Here, α is a constant, which, for now we shall take to be of order unity. We shall examine later, how well such a representation serves to capture the measured experimental mobility and if the constant should be adjusted in any way.

A transport equation for entropy, s_e, is used to close the set of equations for the electron fluid. This equation can be expressed as[12, 13]:

$$\frac{\partial(n_e s_e)}{\partial t} + \nabla \cdot n_e s_e \vec{u}_e = -\nabla \cdot \frac{\vec{q}_e}{T_e} + \dot{s}_e \tag{3}$$

Or, in terms of the substantial derivative:

$$n_e \frac{Ds_e}{Dt} = -\nabla \cdot \frac{\vec{q}_e}{T_e} + \dot{s}_e - \dot{n}_e s_e \tag{4}$$

Here, we have introduced the electron temperature, T_e, and the net rate of electron generation, $\dot{n}_e$. In Eqn. 4, we have the electron heat flux as representing that associated with conduction through the electron fluid, and we lump all other terms (that entropy production due to electron collisions associated with Joule heating, ionization, wall losses) as sources of production.

The electron heat flux term,

$$\vec{q}_e = -K_{eff} \nabla T \tag{5}$$

is expressed in terms of the effective electron thermal conductivity:

$$K_{eff} = \frac{8 n_e k_b^2 T_e \mu_{eff} B \omega_{ce}}{m_e \omega_c^{\,2} \pi} \tag{6}$$

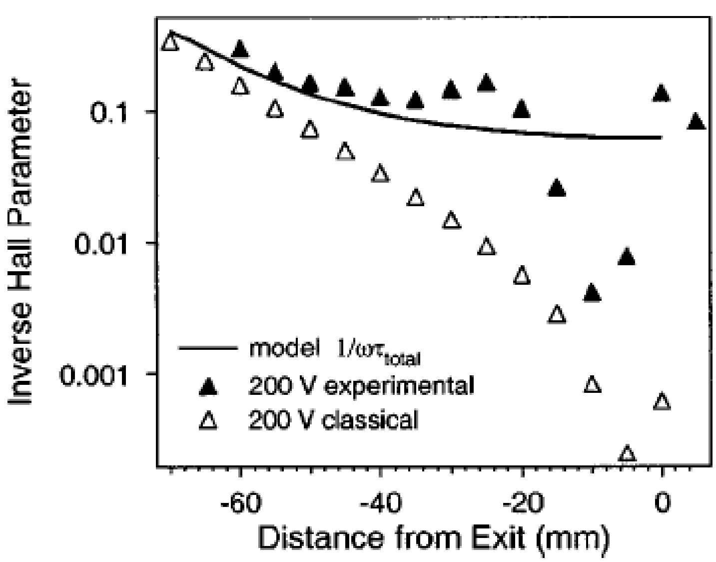

Figure 1. Inverse Hall parameter measured in a laboratory Hall thruster (solid triangles) operating on xenon at 200V (from Ref. 11). The solid line is based on a model that accounts for transport using a Bohm description. The classical data (open triangles) are based on measurements and use of a classical collisional description for the electrons. Note that the Bohm model does not capture the transport barrier seen near the exit plane.

which we can simplify by introducing the mean electron speed, c_e:

$$K_{eff} = n_e k_b \mu_{eff} B \omega_c c_e^2 \frac{1}{\omega_{ce}^{\ 2}} \tag{7}$$

In Eqns. 6 and 7, the effective electron mobility, $\mu_{eff} = \nu_{eff} / (B\omega_{ce})$. The term:

$$-\nabla \cdot \frac{\vec{q}_e}{T_e} = -\frac{1}{T_e}\nabla \cdot \vec{q}_e + \frac{\vec{q}_e \cdot \nabla T_e}{T_e^{\ 2}} = \frac{1}{T_e}\nabla \cdot K_{eff}\nabla T_e - \frac{K_{eff}(\nabla T_e)^2}{T_e^{\ 2}} \approx -K_{eff}\left(\frac{\partial \ln T_e}{\partial n_\perp}\right)^2 \tag{8}$$

By substitution of Eqn. 7 and 8 into Eqn. 4, the entropy equation reduces to a first-order differential equation for the effective mobility. Within the framework of a quasi-1D hybrid simulation[6], this equation describes the spatial evolution in the electron mobility along a direction normal to the magnetic field contours ($n_\perp$):

$$\frac{8}{\pi}\frac{k_b T_e}{e}\frac{\partial \ln T_e}{\partial n_\perp}\frac{d\mu_{eff}}{dn_\perp} + \mu_{eff}^2 \alpha B^3 e / m_e + \beta \mu_{e_{eff}} - \frac{\dot{n}_e s_e}{k_b n_e} = 0 \tag{9}$$

By assuming that the Sackur-Tetrode equation expresses the equilibrium entropy of an ideal electron gas[†],

$$s_e = \frac{5k_b}{2}\ln(T_e) - k_b \ln(n_e k T_e) + k_b\left\{\ln\left[\left(\frac{2\pi m_e}{h^2}\right)^{3/2} k_b^{\ 5/2}\right] + \frac{5}{2}\right\} \tag{10}$$

and that the electron drift velocity along a direction normal to the magnetic field contours is[2]:

$$u_e = -\mu_{e_{eff}}\frac{k_b T_e}{e}\left[\frac{eE}{k_b T_e} + \frac{d}{d\hat{n}_\perp}\ln n_e + \frac{d}{d\hat{n}_\perp}\ln T_e\right] \tag{11}$$

the variable β in Eqn (9) becomes:

$$\beta = \frac{k_b T_e}{e}\left[\frac{eE}{k_b T_e} + \frac{d\ln n_e}{d\hat{n}_\perp} + \frac{d\ln T_e}{d\hat{n}_\perp}\right]\left[\frac{3}{2}\frac{d\ln T_e}{d\hat{n}_\perp} - \frac{d\ln n_e}{d\hat{n}_\perp}\right] + \frac{8}{\pi}\frac{k_b T_e}{e}\frac{d\ln T_e}{dn_\perp}\frac{\partial \ln n_e}{\partial n_\perp} + \frac{8}{\pi}\frac{k_b T_e}{e}\left(\left(\frac{d\ln T_e}{dn_\perp}\right)^2 + \frac{d^2 \ln T_e}{dn_\perp^2}\right) \tag{12}$$

The use of Eqn. 9 adds another equation to the usual set of equations solved in Hall thruster models. It is apparent that the solution to Eqn. 9 is particularly challenging because it is singular. The coefficient of the first derivative term has a zero crossing where the electron temperature is a maximum inside the discharge channel. In our first analyses, we evaluate the significance of this leading term, and (as described below), it is found to not be strongly contributing. We believe therefore, that, without loss of accuracy a reduced equation which is quadratic in form is an adequate representation of the spatial variation in the mobility:

$$\frac{8}{\pi}\mu_{eff}^2 B^3 e / m_e + \beta \mu_{e_{eff}} - \frac{\dot{n}_e s_e}{k_b n_e} = 0 \tag{13}$$

In this paper, we first evaluate the performance of this model using the time-average centerline simulations from our hybrid model[6] of a laboratory Hall thruster[11]. This simulation uses an experimentally-developed electron mobility[11] for a 200V operating condition, which reproduces the experimentally-measured plasma parameters reasonably well.

[†]Assuming equilibrium for the local entropy implies that departures from equilibrium are not too severe.

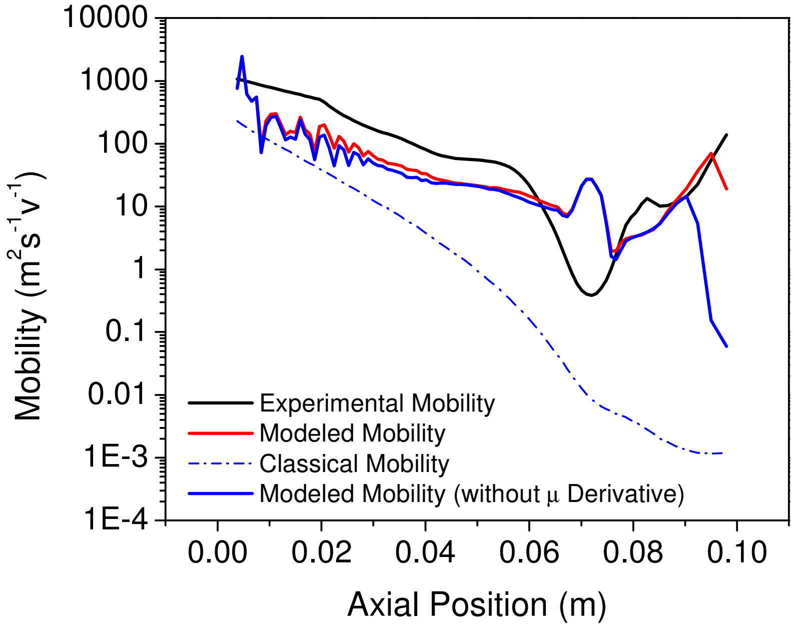

Figure 2. Comparison of model to experimental and classical mobility for the 200V discharge case. The modeled mobility includes the term proportional to the derivative of the mobility by using the experimental mobility. Also shown is the case without this derivative term. It is apparent that the derivative term does not make a large difference in the modeled mobility.

III. Results

As a first test, we assume that the derivative term in Eqn. (9) is small, and solve the resulting quadratic equation for μ_{eff}. The results of this calculation (solid blue line) are compared to the expected classical (dashed blue line) and experimentally-measured mobility (solid line) in Fig.2 for the 200V Hall discharge described in Ref. 11. The results are quite encouraging in that they track the general trends seen in the experiments, but we see that the results do not quite capture the strong transport barrier. It is noteworthy however; that the uncertainty in the experimental measurements of the mobility spans an order of magnitude[11] and so this model is within reasonable range of the measured experimental mobility. The relatively good agreement suggests that it is not necessary to adjust the parameter, α, in the production model, considering the levels of uncertainty noted in the experiments. Wee see that the resulting computed mobility approaches the classical value upstream towards the anode. In regions of the maximum in the magnetic field (i.e., near the exit plane, at a position of z = 0), the mobility is several orders of magnitude higher than this classical value, i.e., the model captures the anomalously high mobility in this region.

To avoid solving directly the differential equation but to evaluate the importance of the derivative term in Eqn. 9, we re-evaluate the quadratic expression with the derivative term determined using the experimental mobility, μ_{effex}. These results are also plotted in Fig. 2 as the solid red line. We see that this approximation leads to a result that is not significantly different than that generated by setting this term equal to zero with the exception perhaps in the region well beyond the exit plane. Note once again that the model does not seem to produce a strong transport barrier in regions of the thruster (just upstream of the exit plane) where experiments seem to suggest that there should be. Despite this, we show in an accompanying paper[14], that the modeled mobility does a good job at capturing the plasma property variations measured along the discharge channel. If this model is found to be generally applicable for determining electron transport characteristics, then it affords much convenience as it leads to an algebraic equation that adds little computational burden on a hybrid simulation and we avoid having to deal with the singularity associated with the zero crossing in temperature.

A second test of this model is to see how it performs at determining the mobility in a more widely-studied Hall thruster, the SPT-100. We use the estimated centerline properties measured by Bishaev and Kim[15] to model the mobility, and compare these to the experimental mobility that is extracted from these properties by Fife[10]. Figure 3 compares this experimental mobility (solid red line) to the mobility as predicted by the model using a entropy-

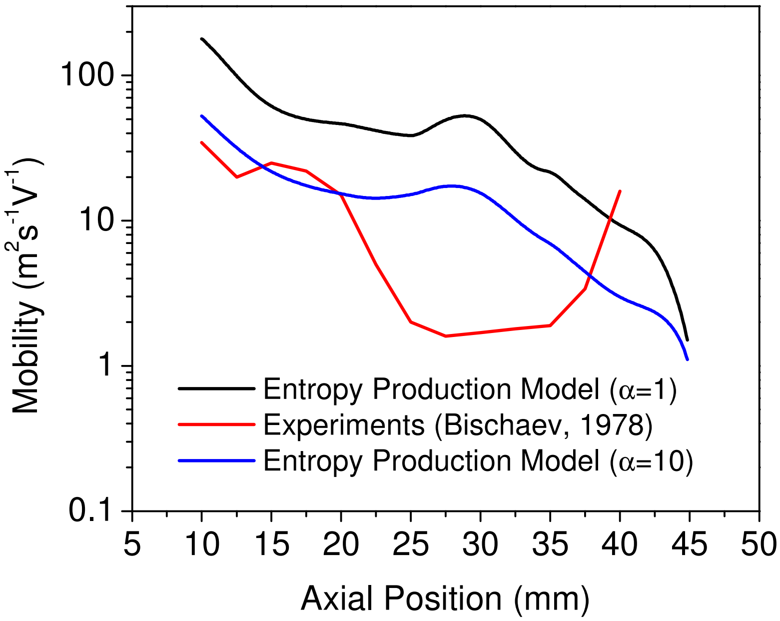

Figure 3. Comparison of model to experimental mobility for the SPT-100 studies of Ref. 15. The solid red line is the mobility as estimated by Fife (Ref. 10) using the data of Ref. 15. The solid black line is the modeled mobility using the baseline value for the entropy production parameter $\alpha = 1$. The solid blue line is the modeled mobility with the parameter $\alpha = 10$.

production factor $\alpha = 1$ (solid black line). The model seems to over predict the mobility throughout the channel, although we emphasize that the uncertainties in Ref. 13 are not reported, and that the experimental mobility is synthesized from several independent measurements of plasma parameters, each of which can have systematic experimental uncertainties that can result in an overall level of uncertainty that is greater than an order of magnitude.[11] Despite this, we did examine what adjustment to the parameter a is needed to bring the model closer to the experimental results. For comparison, we show the model calculation based on a parameter value $\alpha = 10$ (solid blue line), which seems to fall closer to the measurements.

IV. Summary

In this paper, we have presented a new method for independently determining the electron mobility in a Hall thruster. Our approach is based on modeling the electron entropy production. We postulate that the rate of entropy production depends on certain plasma parameters and use dimensional analysis to identify the relevant scaled variables, one of which is the effective inverse Hall parameter. The rate of entropy production is a critical component of the entropy transport equation, which, for quasi-one dimensional problems becomes a first-order differential equation for the effective electron mobility. This additional equation closes the set of equations for the electron fluid. The electron mobility therefore becomes a calculated parameter in simulations. Results are presented for the modeled electron mobility in a laboratory Hall thruster and an SPT-100. While encouraging, the model does rely on experiments to determine the coefficient, α, that describes the functional dependence of the scaled entropy production on the inverse Hall parameter. In studies of a laboratory Hall thruster, this parameter appears to be of order unity, however, experiments in an SPT thruster suggests that this parameter might be somewhat higher. More experiments are needed to further determine the value of this parameter, and its transportability across thruster geometries and operating regimes and conditions.

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